

The Connectivity and the Harary Index of a Graph*

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Abstract The Harary index of a graph is defined as the sum of reciprocals of distances between all pairs of vertices of the graph. In this paper we provide an upper bound of the Harary index in terms of the vertex or edge connectivity of a graph. We characterize the unique graph with maximum Harary index among all graphs with given number of cut vertices or vertex connectivity or edge connectivity. In addition we also characterize the extremal graphs with the second maximum Harary index among the graphs with given vertex connectivity.

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1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *distance* between two vertices u, v of G , denoted by $d_G(u, v)$, is defined as the minimum length of the paths between u and v in G . The Harary index of a graph G , denoted by $H(G)$, has been introduced independently by Plavšić et al. [9] and by Ivanciuc et al. [7] in 1993 for the characterization of molecular graphs. It has been named in honor of Professor Frank Harary on the occasion of his 70th birthday. The *Harary index* $H(G)$ is defined as the sum of reciprocals of distances between all pairs of vertices of the graph G , i.e.

$$H(G) = \sum_{u, v \in V(G)} \frac{1}{d_G(u, v)}.$$

Mathematical properties and applications of the Harary index are reported in [2, 3, 5, 8, 16]. Note that in any disconnected graph G , the distance is infinite between any two vertices from two distinct components. Therefore its reciprocal can be viewed as 0. Thus, we can define validly the Harary index of disconnected graph G as follows:

$$H(G) = \sum_{i=1}^k H(G_i),$$

where G_1, G_2, \dots, G_k are the components of G .

Another distance-based topological index of a graph G is the Wiener index, denoted by $W(G)$. As an oldest topological index, the *Wiener index* of a graph G , first introduced by Wiener [10] in 1947, was defined as

$$W(G) = \sum_{u, v \in V(G)} d_G(u, v).$$

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The motivation for introduction of the Harary index was pragmatic - the aim was to design a distance index differing from the Wiener index in that the contributions to it from the distant atoms in a molecule should be much smaller than from near atoms, since in many instances the distant atoms influence each other much less than near atoms.

Let $\gamma(G, k)$ be the number of vertex pairs of the graph G that are at distance k . Then

$$H(G) = \sum_{k \geq 1} \frac{1}{k} \gamma(G, k). \quad (1.1)$$

It will be convenient to determine the exact value by Eq. (1.1) for some graphs with simple structure (e.g. the graphs with small diameter), but in general it is very difficult to give the exact value of $\gamma(G, k)$. So it is very useful to provide upper or lower bounds for the Harary index; see e.g. [1, 5, 16]. In addition, the extremal Harary index of a given class of graphs has also been studied extensively; see e.g. [4, 6, 11, 12, 13, 14, 15].

In this paper we provide an upper bound of the Harary index in terms of the vertex or edge connectivity of a graph. We characterize the unique graph with maximum Harary index among all graphs with given number of cut vertices or vertex connectivity or edge connectivity. In addition we also characterize the extremal graphs with the second maximum Harary index among the graphs with given vertex connectivity.

2 Main results

In Section 2.1 we determine the unique graph with maximum Harary index among all graphs with given number of cut vertices. We find the optimal graph is surely connected with vertex or edge connectivity 1. In Section 2.2 we consider a general problem, that is, determining the graph(s) with maximum Harary index among all graphs with fixed vertex or edge connectivity. By these results we provide an upper bound of Harary index of a graph in terms of the vertex or edge connectivity.

We introduce some notions used in this paper. Let G be a graph. For a vertex $v \in V(G)$, denote by $N_G(v)$ the neighborhood of v in G and by $d_G(v) = |N_G(v)|$ the degree of v in G . A vertex of G is called *pendent* if it has degree 1, and the edge incident with a pendent vertex is a *pendent edge*. A *pendent path* at v in a graph G is a path in which no vertex other than v is incident with any edge of G outside the path, where the degree of v is at least three. A *cut vertex* of a graph is a vertex whose removal increases the number of components of the graph. A *block* of a connected graph is defined to be a maximum connected subgraph without cut vertices. The *vertex connectivity* (respectively, *edge connectivity*) of a graph, is the minimum number of vertices (respectively, minimum number of edges) whose deletion yields the resulting graph disconnected or a singleton.

For a subset $W \subset V(G)$, let $G - W$ be the subgraph of G obtained by deleting the vertices of W together with the edges incident with them. Similarly, for a subset $E_1 \subset E(G)$, denote by $G - E_1$ the subgraph of G obtained by deleting the edges of E_1 . For an edge set $E_2 \not\subset E(G)$, if two endpoints of any edge in E_2 belong to $V(G)$, then we denote by $G + E_2$ the graph obtained from G by adding the edges of E_2 . Denote by $P_n = P_{v_1 v_2 \cdots v_n}$ a path on vertices v_1, v_2, \dots, v_n with edges $v_i v_{i+1}$ for $i = 1, 2, \dots, n-1$; and denote by K_n the complete graph on n vertices.

2.1 Maximum Harary index with given number of cut vertices

LEMMA 2.1 [12] *Let G be a graph with $u, v \in V(G)$. If $uv \notin E(G)$, then $H(G) < H(G + uv)$. If $uv \in E(G)$, then $H(G) > H(G - uv)$.*

LEMMA 2.2 *Let G_1, G_2, P_s be pairwise vertex-disjoint connected graphs, where G_1 contains an edge uv such that $N_{G_1}(u) \setminus \{v\} = N_{G_1}(v) \setminus \{u\} = \{w_1, w_2, \dots, w_k\}$ ($k \geq 1$), G_2 contains a shortest path*

$Px_1 \cdots x_t$ ($t \geq s+2$) from x_1 to x_t , and $P_s = Pz_1z_2 \dots z_s$. Let G be obtained from G_1 by identifying u with x_1 of G_2 and identifying v with z_1 of P_s , and let $G' = G - \{vw_1, vw_2, \dots, vw_k\} + \{x_2w_1, x_2w_2, \dots, x_2w_k\}$. Then

$$H(G) < H(G'),$$

where the graphs G and G' are shown in Fig. 2.1.

Proof: Let P be the path of G obtained by connecting the paths $Px_1 \cdots x_t$, Puv and $Pz_1z_2 \dots z_s$, where $u = x_1$ and $v = z_1$. Partition the vertex set of G as

$$V(G) = (V(G_1) \setminus \{u, v\}) \cup (V(G_2) \setminus \{x_i : i = 1, 2, t\}) \cup V(P) =: S_1 \cup S_2 \cup S_3.$$

From G to G' , the distance between any two vertices in each S_i is unchanged for $i = 1, 2, 3$; the distance from any vertex of S_1 to any of S_2 is not increased; the distance from any vertex of S_1 to any of z_i ($i = 1, \dots, s$) of S_3 is increased by 1, and to any of x_i ($i = 2, \dots, t$) are decreased by 1, and to the vertex u is unchanged; the distance from any vertex of S_2 and any of S_3 is unchanged.

For any vertex $y \in S_1$, assuming that $d_{G'}(y, z_1) = a$ (≥ 2), then $d_G(y, z_1) = a - 1$, $d_{G'}(y, x_2) = a - 1$, $d_G(y, x_2) = a$. Thus

$$\begin{aligned} \Delta(y) &:= \sum_{i=1}^s \frac{1}{d_{G''}(y, z_i)} + \sum_{i=2}^t \frac{1}{d_{G''}(y, x_i)} - \sum_{i=1}^s \frac{1}{d_{G'}(y, z_i)} - \sum_{i=2}^t \frac{1}{d_{G'}(y, x_i)} \\ &= \sum_{i=0}^{s-1} \frac{1}{a+i} + \sum_{i=0}^{t-2} \frac{1}{a-1+i} - \sum_{i=0}^{s-1} \frac{1}{a-1+i} - \sum_{i=0}^{t-2} \frac{1}{a+i} \\ &= \sum_{i=s}^{t-2} \left(\frac{1}{a-1+i} - \frac{1}{a+i} \right) \\ &> 0. \end{aligned}$$

So

$$H(G') - H(G) \geq \sum_{y \in S_1} \Delta(y) > 0.$$

The result follows. ■

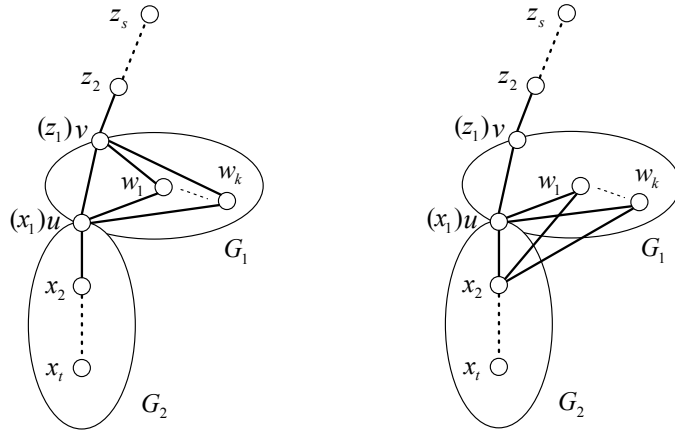


Fig. 2.1. The graphs G (left) and G' (right) in Lemma 2.2.

REMARK 2.3 The graphs G and G' in Lemma 2.2 possess the same number of cut vertices. In addition, if taking $s = 1$ (i.e. the path attaching at v is trivial), the edge uv of G will become a pendant edge of G' .

If taking $G_2 = Px_1 \cdots x_t$ in Lemma 2.2, we will have the result below, which has been shown in [12] and [5] under a more general condition.

COROLLARY 2.4 *Let G be a connected graph containing an edge uv such that $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\} \neq \emptyset$. Let $G_{t,s}$ be obtained from G by attaching a path P_t at u and a path P_s at v . If $t \geq s + 2 \geq 3$, then $H(G_{t,s}) < H(G_{t-1,s+1})$.*

LEMMA 2.5 *Let $K_p u K_q$ be the union of two complete graphs K_p, K_q sharing exactly one common vertex u , where $p \geq 3, q \geq 3$. Let G be obtained from $K_p u K_q$ by attaching a path P_t at some vertex $w_1 \in V(K_p) \setminus \{u\}$ and a path P_s at some vertex $v_1 \in V(K_q) \setminus \{u\}$, and possibly attaching some connected graphs at other vertices of $V(K_p u K_q) \setminus \{u, v_1, w_1\}$, where $t \geq s \geq 1$; and let G' be obtained from G by deleting the edges of K_q incident to v_1 except $v_1 u$ and adding all possible edges between each of $V(K_q) \setminus \{v_1\}$ and each of $V(K_p)$; see Fig. 2.2 for the graph G and G' . Then $H(G) < H(G')$.*

Proof: Let G^* be the component of $G - \{u, v_1\}$ which contains the vertices of K_q , and G^{**} be the component of $G - \{u, w_1\}$ which contains the vertices of K_p . Let $P_s = Pv_1 v_2 \cdots v_s$, $P_t = Pw_1 w_2 \cdots w_t$. Partition the vertex of G as

$$V(G^*) \cup V(P_s) \cup \{u\} \cup V(P_t) \cup V(G^{**}) =: S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5.$$

Observe the transformation from G to G' , the distances between any vertex of S_1 and any of S_2 are increased by 1, the distances between any vertex of S_1 and S_4 are decreased by 1, the distance between any vertex of S_1 and any of S_5 is decreased by 1, and the distance between any other two vertices is not changed.

For any vertex $y \in S_1$, assuming that $d_{G'}(y, v_1) = a (\geq 2)$, then $d_{G'}(y, w_1) = a - 1, d_G(y, v_1) = a - 1, d_G(y, w_1) = a$. Thus

$$\begin{aligned} \Delta(y) &:= \sum_{i=1}^s \frac{1}{d_{G'}(y, v_i)} + \sum_{i=1}^t \frac{1}{d_{G'}(y, w_i)} - \sum_{i=1}^s \frac{1}{d_G(y, v_i)} - \sum_{i=1}^t \frac{1}{d_G(y, w_i)} \\ &= \sum_{i=0}^{s-1} \frac{1}{a+i} + \sum_{i=0}^{t-1} \frac{1}{a-1+i} - \sum_{i=0}^{s-1} \frac{1}{a-1+i} - \sum_{i=0}^{t-1} \frac{1}{a+i} \\ &= \sum_{i=s}^{t-1} \left(\frac{1}{a-1+i} - \frac{1}{a+i} \right) \geq 0. \end{aligned}$$

So,

$$H(G') - H(G) = \sum_{y \in S_1} \Delta(y) + \sum_{(y,z) \in S_1 \times S_5} \left(\frac{1}{d_{G'}(y,z)} - \frac{1}{d_G(y,z)} \right) > \sum_{y \in S_1} \Delta(y) \geq 0.$$

The result follows. ■

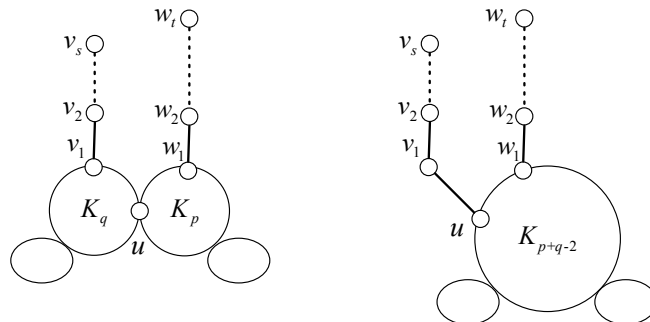


Fig. 2.2. The graphs G (left) and G' (right) in Lemma 2.5

REMARK 2.6 *The graphs G and G' in Lemma 2.5 possess the same number of cut vertices. In addition, if taking $s = 1$, the edge uv_1 of G becomes a pendant edge of G' .*

THEOREM 2.7 *Among all graphs with n vertices and k cut vertices, where $0 \leq k \leq n - 2$, the maximal Harary index is attained uniquely at the graph $\mathbf{G}_{n,k}$, where $\mathbf{G}_{n,k}$ is obtained from K_{n-k} by attaching $n - k$ paths of almost equal lengths to its vertices respectively.*

Proof: Let G be a graph with the maximal Harary index among all the graphs with n vertices and k cut vertices. If $k = 0$, then by Lemma 2.1, $G = K_n = \mathbf{G}_{n,0}$. Suppose in the following that $1 \leq k \leq n - 2$. The result will hold by the following claims.

Claim 1: G is connected. Assume that G is disconnected. Let z be a cut vertex of G . Then z is also a cut vertex of some component, say G_1 , of G . Let G_2 be a component of G different from G_1 . If there is a cut vertex, say z' , in G_2 , then $G + zz'$ possesses the same number of cut vertices as G , and by Lemma 2.1, $H(G) < H(G + zz')$, a contradiction. If there are no cut vertices in G_2 , adding edges between z and all vertices of G_2 , we will arrive at a new graph G' , which possesses the same number of cut vertices as G . However, by Lemma 2.1, $H(G) < H(G')$, a contradiction again. So G is connected.

By Lemma 2.1, each block of G is complete, and each cut vertex of G is contained in exactly two blocks. If every block of G is trivial (containing exactly two vertices), i.e., every block is a single edge, then G is a tree with maximum degree two, i.e., $G = P_n = \mathbf{G}_{n,n-2}$. Suppose in the following that G contains nontrivial blocks (on at least three vertices).

Claim 2: If $G \neq \mathbf{G}_{n,1}$, then each pendent block (i.e., the block containing only one cut vertex of G) is an edge. Assume to the contrary, B_1 is a nontrivial pendent block of G . Let u be a vertex of B_1 different from the unique cut vertex, say w , contained in B_1 . Let B_2 be the block adjacent to B_1 . Deleting all edges in B_1 incident to u except uw , and adding all edges between the vertices of $V(B_1) \setminus \{u\}$ and the vertices of $V(B_2)$, we obtain a new graph G' with the same number of cut vertices as G . By Remark 2.3 and Remark 2.6, and the fact that $G \neq \mathbf{G}_{n,1}$, we have $H(G) < H(G')$, a contradiction.

Choose a pendent path, say P_s attached at v of some nontrivial block B , whose length is minimum among all pendent paths of G . We stress that P_s may be trivial, i.e. $s = 1$ or P_s contains only the vertex v .

Claim 3: The component attached at any vertex of B is a path (possibly being trivial). For $x \in V(B)$, let $H^{(x)}$ be the component of $G - E(B)$ containing x . Obviously, $H^{(v)} = P_s$. Suppose u is an arbitrary vertex of B and $u \neq v$. Obviously, $N_B(v) \setminus \{u\} = N_B(u) \setminus \{v\}$. Let G_1 be the component of $G - E(H^{(u)} \cup E(P_s))$ containing u , which surely contains the block B .

Assume that $H^{(u)}$ is not a (possibly trivial) path. Then $H^{(u)}$ contains a nontrivial block. By the proof of Claim 2, $H^{(u)}$ must contain a nontrivial pendent path P_t attached at some nontrivial block B' of $H^{(u)}$, where $t \geq s$. So $H^{(u)}$ contains a shortest path P_r from u to the pendant vertex of P_t , where $r \geq 3$ and $r \geq t + 1 \geq s + 1$. If $r \geq s + 2$ or $s = 1$, by Lemma 2.2 and Remark 2.3, we may get another graph with n vertex and k cut vertices, which has a larger Harary index, a contradiction. So, it suffices to consider the case: $s > 1$, B' shares with G_1 (also the block B) the common vertex u , and $H^{(u)}$ is obtained from B' by attaching P_s at each of its vertices except u . Now applying Lemma 2.5, we may get a new graph with n vertices and k cut vertices, which has a larger Harary index, a contradiction. Hence $H^{(u)}$ is a pendent path attached at u which contains at least s vertices.

Claim 4: All paths attached at the vertices of B have almost equal lengths. This can be shown by Corollary 2.4. ■

THEOREM 2.8

$$H(K_n) = H(\mathbf{G}_{n,0}) > H(\mathbf{G}_{n,1}) > H(\mathbf{G}_{n,2}) > \cdots > H(\mathbf{G}_{n,n-2}) = H(P_n).$$

Furthermore, if a graph G of order $n \geq 3$ contains cut vertices or cut edges, then

$$H(G) \leq H(\mathbf{G}_{n,1}),$$

with equality if and only if $G = \mathbf{G}_{n,1}$.

Proof: Let $G = \mathbf{G}_{n,k}$, where $k \geq 1$. Let P_s be a pendant path of G attached at u with maximum length. Surely $s \geq 2$. Let v be the vertex on the path P_s adjacent to u . Adding all possible edges between v and the vertices of K_{n-k} (the subgraph of G), we will arrive at a graph $G' = \mathbf{G}_{n,k-1}$, which holds that $H(\mathbf{G}_{n,k-1}) > H(G) = H(\mathbf{G}_{n,k})$ by Lemma 2.1. The first assertion follows. The remaining parts of this theorem can be obtained from above discussion and Theorem 2.7. ■

2.2 Maximum Harary indices with given connectivity and edge connectivity

In Section 2.1 we have determined the unique graph with maximum Harary index among all graphs with vertex or edge connectivity 1; see Theorem 2.8. Now we consider a general problem, i.e. characterizing the graph(s) with maximum Harary index among all graphs with fixed vertex or edge connectivity k .

We first give some notations. For two vertex-disjoint graphs G and H , let $G \cup H$ denote the union of G and H , and $G \vee H$ denote the graph obtained from $G \cup H$ by adding all possible edges between the vertices of G and the vertices of H . Denote $G_{n_1, n_2, n_3} := (K_{n_1} \cup K_{n_2}) \vee K_{n_3}$, where $n_1 \geq n_2 \geq 1$ and $n_3 \geq 1$. Denote by \mathcal{G}_n^r (respectively, $\overline{\mathcal{G}}_n^r$) the set of all connected graphs of order n with vertex connectivity r (respectively, edge connectivity r). Clearly, $1 \leq r \leq n-1$, and $\mathcal{G}_n^{n-1} = \overline{\mathcal{G}}_n^{n-1} = \{K_n\}$. So it is enough to consider the case of $1 \leq r \leq n-2$. Let $K(n-1, r)$ be a graph obtained from K_{n-1} by adding a vertex together with edges joining this vertex to r vertices of K_{n-1} , where $1 \leq r \leq n-2$. Surely $K(n-1, r) \in \mathcal{G}_n^r$ and $K(n-1, r) \in \overline{\mathcal{G}}_n^r$.

LEMMA 2.9 *If $n_1 \geq n_2 \geq 2$ and $n_3 \geq 1$, then $H(G_{n_1, n_2, n_3}) < H(G_{n_1+1, n_2-1, n_3})$.*

Proof: Observe that the graph G_{n_1, n_2, n_3} can be considered as one obtained from G_{n_1, n_2-1, n_3} by adding a vertex, say u , and connecting u with all vertices of $K_{n_2-1} \cup K_{n_3}$, and G_{n_1+1, n_2-1, n_3} can be considered as one also obtained from G_{n_1, n_2-1, n_3} by adding a vertex, also say u for simplicity, and connecting u with all vertices of $K_{n_1} \cup K_{n_3}$. So, from G_{n_1, n_2, n_3} to G_{n_1+1, n_2-1, n_3} , the distance between u and any vertex of K_{n_1} is decreased by 1, the distance between u and any vertex of K_{n_2-1} is increased by 1, and the distance between any other two vertices is unchanged. Therefore,

$$H(G_{n_1, n_2, n_3}) - H(G_{n_1+1, n_2-1, n_3}) = \left(\frac{n_1}{2} + n_2 - 1\right) - \left(n_1 + \frac{n_2 - 1}{2}\right) = \frac{n_2 - n_1 - 1}{2} < 0.$$

The result follows. ■

THEOREM 2.10 *For each $r = 1, 2, \dots, n-2$, the graph $K(n-1, r)$ is the unique one with the maximum Harary index among all graphs of order n and vertex connectivity r .*

Proof: Let G be a graph that attains the maximum Harary index in \mathcal{G}_n^r . Let U be a vertex cut of G containing r vertices such that $G - U$ has components G_1, G_2, \dots, G_s , where $s \geq 2$. Firstly, we assert that $s = 2$; otherwise adding all possible edges within the graph $G_1 \cup G_2 \cup \dots \cup G_{s-1}$, we would get a graph belonging to \mathcal{G}_n^r but with a larger Harary index. Similarly, the induced subgraph $G[U]$, and the subgraphs G_1, G_2 are all complete, and each vertex of U joins all vertices of G_1 and G_2 . Without loss of generality, we assume that $|V(G_1)| =: n_1 \geq |V(G_2)| =: n_2$. By Lemma 2.9, $n_2 = 1$, and hence $G = K(n-1, r)$. ■

THEOREM 2.11 *For each $r = 1, 2, \dots, n-2$, the graph $K(n-1, r)$ is the unique one with the maximum Harary index among all graphs of order n and edge connectivity r .*

Proof: Let G be a graph that attains the maximum Harary index in $\overline{\mathcal{G}}_n^r$. Assume the vertex connectivity of G is r_0 . Then $r_0 \leq r$. So

$$H(G) \leq H(K(n-1, r_0)) \leq H(K(n-1, r)) \leq H(G),$$

where the first inequality holds by Theorem 2.10, the second equality holds by Lemma 2.1 as $K(n-1, r)$ is obtained from $K(n-1, r_0)$ by adding $r - r_0$ edges, and the last inequality holds as $K(n-1, r) \in \overline{\mathcal{G}}_n^r$. Hence, all inequalities above become equalities, which implies $r = r_0$, and $G = K(n-1, r)$ from the first equality by Theorem 2.10. \blacksquare

COROLLARY 2.12 *Let G be a graph of order n with vertex or edge connectivity r , where $1 \leq r \leq n-2$. Then*

$$H(G) \leq \frac{(n-1)^2 + r}{2},$$

with equality holds if and only if $G = K(n-1, r)$.

Proof: By Theorems 2.10 and 2.11, we only need to calculate the Harary index of $K(n-1, r)$. Since $K(n-1, r)$ is a graph with diameter 2, the number of pairs of vertices with distance 1 is $C_{n-1}^2 + r$, and the number of pairs of vertices with distance 2 is $n - r - 1$, we have

$$H(K(n-1, r)) = C_{n-1}^2 + r + \frac{n-r-1}{2} = \frac{(n-1)^2 + r}{2}.$$

Finally, we characterize the graphs with the second maximum Harary index among all the graphs of order n with vertex connectivity r . \blacksquare

THEOREM 2.13 *Let G be a graph with the second maximum Harary index among all graphs of order n and vertex connectivity r , where $1 \leq r \leq n-2$.*

- (1) *If $r = n-2$, then G is obtained from $K_{n-2} \vee O_2$ by deleting an arbitrary edge in K_{n-2} .*
- (2) *If $1 \leq r \leq n-3$ and $r \neq n-4$, then G is a graph obtained from $K(n-1, r)$ by deleting an arbitrary edge in the induced subgraph K_{n-1} .*
- (2) *If $r = n-4$, then $G = G_{2,2,n-4}$ or G is obtained from $K(n-1, r)$ by deleting an arbitrary edge in the induced subgraph K_{n-1} .*

Proof: Let U be a vertex cut of G containing r vertices such that $G-U$ has components G_1, G_2, \dots, G_s , where $|V(G_1)| \geq |V(G_2)| \geq \dots \geq |V(G_s)|$, and $s \geq 2$. If $r = n-2$, then $s = 2$, and G is obtained from $K(n-1, n-2) = K_{n-2} \vee O_2$ by deleting an arbitrary edge in K_{n-2} . So we assume $r \leq n-3$.

We first claim that $s = 2$, or $s = 3$ and $|V(G_1)| = |V(G_2)| = |V(G_3)| = 1$. Otherwise, $s \geq 4$. By connecting one vertex of G_1 with one of G_2 , we will arrive at a new graph G' . Obviously, $G' \in \mathcal{G}_n^r$ but $G' \neq K(n-1, r)$; and by Theorem 2.10, $H(K(n-1, r)) > H(G') > H(G)$, a contradiction. If $s = 3$ and $|V(G_1)| \geq 2$. Then we attain a new graph G'' by connecting one vertex of G_1 with one of G_2 . Similarly we also have $H(K(n-1, r)) > H(G'') > H(G)$, a contradiction.

If $s = 3$ and G_1, G_2, G_3 are all single points, then $G = O_3 \vee K_{n-3}$. Now suppose $s = 2$. By Lemma 2.9, if $|V(G_2)| = 1$, then $G \in \{G_{n-r-1,1,r} - e_1, G_{n-r-1,1,r} - e_3, G_{n-r-1,1,r} - e_{13}\}$; if $|V(G_2)| > 1$, then $r \leq n-4$ and $G = G_{n-r-2,2,r}$; where e_1, e_3 are respectively the (arbitrary) edges in K_{n-r-1}, K_r and e_{13} is an (arbitrary) edges connecting K_{n-r-1} and K_r , by recalling $G_{n-r-1,1,r} = (K_{n-r-1} \cup K_1) \vee K_r$. Observe that $O_3 \vee K_{n-3} = G_{n-r-1,1,r} - e_1$, where $r = n-3$.

By a little calculation,

$$H(G_{n-r-1,1,r} - e_1) = H(G_{n-r-1,1,r} - e_3) = H(G_{n-r-1,1,r} - e_{13}) = \frac{n^2 - 2n + r}{2}$$

and when $r \leq n - 4$

$$H(G_{n-r-2,2,r}) = \frac{n^2 - 3n + 2r + 4}{2}.$$

If $1 \leq r < n - 4$, $\frac{n^2-2n+r}{2} > \frac{n^2-3n+2r+4}{2}$; if $r = n - 4$, $\frac{n^2-2n+r}{2} = \frac{n^2-3n+2r+4}{2}$. So, if $1 \leq r \leq n - 3$ and $r \neq n - 4$, then G is one of $G_{n-r-1,1,r} - e_1$, $G_{n-r-1,1,r} - e_3$ or $G_{n-r-1,1,r} - e_{13}$, namely G is obtained from $K(n - 1, r)$ by deleting an arbitrary edge in the induced subgraph K_{n-1} . If $r = n - 4$, then $G = G_{2,2,n-4}$ or G is obtained from $K(n - 1, r)$ by deleting an arbitrary edge in the induced subgraph K_{n-1} . ■

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